Generalized Goldbach Conjecture and Integer Coverages

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Finite Coverages

2.1 Definitions

Let $\Omega = \{x_1, \cdots, x_n\}$ be a set of positive integers with $0 \leq x_1 < x_2 < \cdots < x_n$.

Definition We define the **closure** of Ω (under the addition operator) as the set

$$\bar{\Omega} = \{x + y \dagger x, y \in \Omega \text{ and } 0 \leq x + y \leq \sup(\Omega)\}$$

To avoid confusion, Ω (resp. $\bar{\Omega}$) is sometimes denoted as Ω_n (resp. $\bar{\Omega}_n$). The definition easily extends to the case $n = \infty$.

Note that as $n \to \infty, x_n \to \infty$ and thus $\bar{\Omega}_n$ becomes an infinite subset of N, or N itself depending on

- How fast the sequence (x_n) grows
- How dense Ω_n is for small values of n
- How randomly distributed the x_n are, particularly for small values of n

Given a sequence (Ω_n), we define Ω_∞ as the limiting set

$$\Omega_\infty = \lim_{n\to\infty} \Omega_n = \cup_{k=1}^{\infty} \Omega_n \tag{2.1}$$

Definition We define the **complete closure** $\bar{\Omega}_\infty$ as

$$\bar{\Omega}_\infty = \lim_{n\to\infty} \bar{\Omega}_n = \cup_{k=1}^{\infty} \bar{\Omega}_n \tag{2.2}$$

We will provide examples of $\bar{\Omega}_\infty$ that fail to cover all the integers because one or more of the three above conditions are not met.

Definition Let V, W be two sets of positive integers. We say that V covers W (or V is a **coverage** of W) if and only if $W \subseteq V$. This definition applies both to finite or infinite sets.We say that V is an **exact coverage** of W if $W = V$.

Additional coverage and closure definitions and concepts will be introduced in the next chapters.

2.2 Combinatorial Results

Theorem 2.2.1 *Let us randomly pick out m objects (e.g. integers) out of a set of n objects, with replacements (thus, m might be greater than n). The expected number of distinct objects $E_{n,m}$ is given by the formula*

$$E_{n,m} = n \cdot \left\{1 - \left(1 - \frac{1}{n}\right)^m\right\} \tag{2.3}$$

Proof The probability that k distinct objects are selected given m drawings with replacement from n objects is

$$P_{n,m}(k) = S_2(n,k) \cdot \frac{n!}{(n-k)!} \cdot n^{-m} \tag{2.4}$$

where $1 \leq k \leq n$ and $S_2(n,k)$ are Stirling numbers of the second kind. For details, see [1]. The expectation is

$$E_{n,m} = n \cdot (1 - (1 - 1/n)^m)$$

and the variance is

$$V_{n,m} = n \cdot (1 - 1/n)^m + n \cdot (n-1) \cdot (1 - 2/n)^m - n^2 \cdot (1 - 1/n)^{2m} \tag{2.5}$$

Proof of expectation is very easy:

- P(particular object not selected in a particular drawing) $= (1 - 1/n)$.
- P(particular object not selected in m drawings) $= (1 - 1/n)^m$.
- P(particular object selected in m drawings) $= (1 - (1 - 1/n)^m)$.
- E(number of distinct objects in m drawings) $= n \cdot (1 - (1 - 1/n)^m)$. ∎

Theorem 2.2.2 *Let $\Omega = \{x_1, \cdots, x_n\}$ be a set of positive real numbers with $0 \leq x_1 < \cdots < x_n$. Let f be an arbitrary strictly monotone and continuous real valued function f defined on R^+, satisfying $x_k = f(k)$ for all integers $k = 1, \cdots, n$. Then*

$$\sum_{\substack{x, y \in \Omega \\ y \geq x}} I(x + y \in \Omega) = \sum_{k=1}^{n} \max\{q(k, n) - k + 1, 0\} \tag{2.6}$$

where

$$q(k, n) = n - f^{-1}(f(n) - f(k)) \tag{2.7}$$

Proof Here $x + y \in \Omega$ if $x + y < \sup \Omega = x_n$. Also note that a strictly monotone continuous function f satisfying the above assumptions always exists.Thus $q(k, n)$ is uniquely defined. [...] $\blacksquare$

2.3 Prime Numbers

In this section, we are concerned with traditional prime numbers defined by the well know sequence $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \cdots\}$. However, none of the unique properties of prime numbers will ever be used in this book. We could have worked with pseudo-primes or other related sequences without changing anything in our text. The only properties of interest are asymptotic behavior, density and randomness for small values of n. For that matter, we could have defined the prime numbers in a way that does not involve any divisibility or congruences, as below.

Theorem 2.3.1 *Prime numbers can be defined without using divisibility nor congruence concepts, but rather by relying on simple continuous trigonometric functions.*

Proof For any positive real z, let

$$\phi(z) = \frac{\pi z}{\sin(\pi z)} \cdot \prod_{k=2}^{\infty} \frac{\sin(\pi z/k)}{\pi z/k}$$

We have

$$\phi(z) = \frac{1}{1 - z^2} \cdot \prod_{k=2}^{\infty} \Big\{ \frac{\sin(\pi z/k)}{\pi z/k} \cdot \frac{1}{1 - z^2/k^2} \Big\}$$

Thus

$$\begin{aligned}
\phi(z) &= 0 \text{ if } z \text{ is composite} \\
\phi(z) &\neq 0 \text{ if } z \text{ is prime} \\
\phi(z+1) &= 0 \text{ if } z \text{ is prime}
\end{aligned}$$

Now let

$$\psi(z) = \frac{\left\{\phi(z+1)\right\}^2}{\lambda\left\{\phi(z)\right\}^{2\rho} + \left\{\phi(z+1)\right\}^2}$$

where $\lambda > 0, \rho > 0$ and $\rho \geq \sqrt{2z}$. When both z and $z+1$ are composite, both the numerator and denominator vanish. In this case, the above expression for $\psi(z)$ should be interpreted as a limit. The exponent ρ guarantees that this limit is equal to 1. Note that $\psi(z)$ is a continuous function for $z > 0$. Its first derivative exists and is finite except when z is an integer. Finally, we have:

$$\begin{aligned}
\psi(z) &= 0 \text{ if } z \text{ is prime} \\
\psi(z) &= 1 \text{ if } z \text{ is composite} \\
\psi(z) &\in \,]0,1[\text{ otherwise}
\end{aligned}$$

Thus identifying all the prime numbers greater than 2 consists of finding all the positive roots of ψ. ∎

2.4 Goldbach Conjecture

2.5 Sums of Squares

2.6 Waring Conjecture

2.7 General Problem

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Chapter 4

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Chapter 6

New Results and Conjectures

Bibliography

[1] Comtet, L. *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. rev. enl. ed. Dordrecht, Netherlands: Reidel, 1974.